

Fixed Point Approach for Complementarity Problems

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In this paper, we use the fixed point technique to suggest a new unified and general algorithm for computing the approximate solution of a nonlinear complementarity problem of finding u such that $u \geq 0$, $Tu + A(u) \geq 0$ ($u, Tu + A(u) = 0$), where T is a continuous mapping from \mathbb{R}^n into itself and A is a non-linear transformation from \mathbb{R}^n into itself. This algorithm includes many existing algorithms for complementarity problems as special cases. Convergence properties are also discussed and analyzed. © 1988 Academic Press, Inc.

1. INTRODUCTION

Complementarity theory as developed by Lemke [1], and Cottle and Dantzig [2] and others in the early 1960s and thereafter, has numerous applications in diverse fields of mathematical and engineering sciences. Recently, complementarity problems have been extended and generalized in various directions to study a wide class of linear and nonlinear problems arising in fluid flow through porous media, contact problems in elasticity, control and optimization, economics and transportation equilibrium, see Crank [3], Cottle [4], Lin and Cryer [5] and the references therein for physical and mathematical formulations.

Inspired and motivated by the recent research work going on in this area, we introduce and study a new class of complementarity problems. We show, using the change of variables, that complementarity problems can be formulated as fixed point problems. This formulation enables us to suggest an iterative method for computing the approximate solution of the complementarity problems. As special cases, we get algorithms for the various known classes of the complementarity problems. These algorithms also appear to be new ones. Most of the convergence properties discussed

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previously for the complementarity problems can be derived as special cases.

In Section 2, we introduce a new class of complementarity problems and discuss various special cases. Using the change of variables, we characterize this class as the fixed point problem. Algorithms and convergence results are discussed and derived in Sections 3 and 4.

2. PRELIMINARIES AND FORMULATIONS

We denote by (\cdot, \cdot) and $\|\cdot\|$, the inner product and norm on \mathbb{R}^n respectively.

Given a continuous mapping T and nonlinear transformation A from \mathbb{R}^n into itself, we consider the problem of finding u such that

$$u \geq 0, \quad Tu + A(u) \geq 0 \quad (u, Tu + A(u)) = 0. \quad (2.1)$$

If T is a nonlinear mapping, then problem (2.1) is known as strongly nonlinear complementarity. If T is an affine transformation of the form $T: u \rightarrow Mu + q$, $M \in \mathbb{R}^{n \times n}$ and $q \in \mathbb{R}^n$, then problem (2.1) is equivalent to

$$u \geq 0, \quad Mu + q + A(u) \geq 0, \quad (u, Mu + q + A(u)) = 0, \quad (2.2)$$

which is called the mildly nonlinear complementarity problem.

Problems (2.1) and (2.2) arise, for example, as finite difference (finite element) approximations to constrained nonlinear partial differential inequalities of the type

$$\begin{aligned} -Lu(\mathbf{x}) + f(\mathbf{x}, u(\mathbf{x})) &\geq 0, & \text{in } D \\ u(\mathbf{x}) &\geq 0, & \text{in } D \\ u(\mathbf{x})[-Lu(\mathbf{x}) + f(\mathbf{x}, u(\mathbf{x}))] &= 0, & \text{in } D \\ u(\mathbf{x}) &= g(\mathbf{x}), & \text{on } S, \end{aligned} \quad (2.3)$$

where L is a given nonlinear (linear) elliptic operator, $D \in \mathbb{R}^n$ is a domain with boundary S , $f(u) \equiv f(\mathbf{x}, u(\mathbf{x}))$ is a nonlinear function of \mathbf{x} and $u(\mathbf{x})$, and g is a given function. Well-known examples of free boundary value problems which can be written in the form (2.3) include fluid flow through porous media, journal bearing lubrication problems, and contact problems in elasticity, see [3–6].

We remark that if the nonlinear transformation A is identically zero, that is $A(u) \equiv 0$, then problems (2.2) and (2.1) are equivalent to find u such that

$$u \geq 0, \quad Mu + q \geq 0 \quad (u, Mu + q) = 0, \quad (2.4)$$

and

$$u \geq 0, \quad Tu \geq 0 \quad (u, Tu) = 0. \quad (2.5)$$

The problem (2.4) is known as the linear complementarity problem, originally introduced and considered by Lemke [1] and Cottle and Dantzig [2]. Iterative methods have been used by many research workers including Mangasarian [7], Cottle [4], Ahn [8, 9], Pang [10], Aganagic [11], Lin and Cryer [5], and Van Bokhoven [15] for solving the linear complementarity problems along with the convergence criteria. While the problem (2.5) is called the nonlinear complementarity problem, see Karamardian [12], Fang [13], and Noor [14].

The problems (2.1) and (2.2) can be written as

$$u \geq 0, \quad v = Mu + q + A(u) \geq 0, \quad (u, v) = 0 \quad (2.6)$$

and

$$u \geq 0, \quad v = Tu + A(u) \geq 0, \quad (u, v) = 0, \quad (2.7)$$

which is useful in developing a fixed point formulation of the complementarity problems.

We consider the following change of variables:

$$u = \frac{|x| + x}{2} \quad (2.8)$$

and

$$v = (\rho E)^{-1} (|x| - x),$$

where $\rho > 0$, and $E \in \mathbb{R}^{n \times n}$ is a positive diagonal matrix. Clearly $u \geq 0$ and $v \geq 0$. Using relations (2.8), we see that the complementarity problems (2.6) and (2.7) are equivalent to finding $x \in \mathbb{R}^n$ such that

$$x = \frac{|x| + x}{2} - \frac{\rho E}{2} \left\{ M \left(\frac{|x| + x}{2} \right) + q + A \left(\frac{|x| + x}{2} \right) \right\} \quad (2.9)$$

and

$$x = \frac{|x| + x}{2} - \frac{\rho}{2} \left\{ T \left(\frac{|x| + x}{2} \right) + A \left(\frac{|x| + x}{2} \right) \right\}, \quad (2.10)$$

respectively, where $\rho > 0$ is a constant.

This formulation of the complementarity problems is by no means an entirely new and novel one, see Van Bokhoven [15] and Noor and Zrae [16].

3. ALGORITHMS

In the previous section, we have shown by using the change of variables (2.8), that the problems (2.6) and (2.7) are equivalent to a fixed point problem of solving

$$x = F(x),$$

where

$$F(x) = \frac{|x| + x}{2} - \frac{\rho E}{2} \left\{ M \left(\frac{|x| + x}{2} \right) + q + A \left(\frac{|x| + x}{2} \right) \right\}$$

or

$$F(x) = \frac{|x| + x}{2} - \frac{\rho}{2} \left\{ T \left(\frac{|x| + x}{2} \right) + A \left(\frac{|x| + x}{2} \right) \right\}$$

for some constant $\rho > 0$ and $E \in \mathbb{R}^{n \times n}$, a positive diagonal matrix.

This alternative formulation is very useful in the approximation and numerical analysis of the complementarity problems. One of the consequences of this formulation is that we can obtain an approximate solution by an iterative scheme.

Based on the above formulation and observations, we now suggest and analyze new general and unified algorithms for problems (2.6) and (2.7) as follows:

ALGORITHM 3.1. Given x_0 , compute x_{n+1} by the iterative scheme

$$\begin{aligned} x_{n+1} = & (1 - \lambda) \frac{|x_n| + x_n}{2} \\ & + \frac{\lambda}{2} \left(|x_n| + x_n - \rho E_n \left\{ M \left(\frac{|x_n| + x_n}{2} \right) + q + L_n (x_{n+1} - x_n) \right. \right. \\ & \left. \left. + A \left(\frac{|x_n| + x_n}{2} \right) \right\} \right), \quad n = 0, 1, 2, \dots, \end{aligned} \quad (3.1)$$

where $\rho > 0$ and $\{E_n\}$ and $\{L_n\}$ are bounded sequences of matrices in $\mathbb{R}^{n \times n}$.

For the above algorithm to be practical, L_n may not be strictly lower or upper triangular matrix, as pointed out in Pang [10]. Here the original data M , remains intact throughout iteration, allowing this algorithm to be efficient both for large scale and specially structured problems.

Although, the matrices E and L are allowed to vary from one iteration

to the next, but for our purpose, we choose E and L to be fixed throughout the algorithm. Thus for $E_n = E$ and $L_n = L$, the Algorithm 3.1 becomes:

ALGORITHM 3.2. For given $x_0 \in \mathbb{R}^n$, compute

$$\begin{aligned} x_{n+1} = & (1 - \lambda) \frac{|x_n| + x_n}{2} \\ & + \frac{\lambda}{2} \left(|x_n| + x_n - \rho E \left\{ M \left(\frac{|x_n| + x_n}{2} \right) + q + L(x_{n+1} - x_n) \right. \right. \\ & \left. \left. + A \left(\frac{|x_n| + x_n}{2} \right) \right\} \right), \quad n = 0, 1, 2, \dots \end{aligned} \quad (3.2)$$

Similarly, for the complementarity problem (2.7), we have the following new, general, and unified algorithm.

ALGORITHM 3.3. For given $x_0 \in \mathbb{R}^n$, compute

$$\begin{aligned} x_{n+1} = & (1 - \lambda) \frac{|x_n| + x_n}{2} + \lambda \left(\frac{|x_n| + x_n}{2} - \frac{\rho}{2} \left\{ T \left(\frac{|x_n| + x_n}{2} \right) \right. \right. \\ & \left. \left. + A \left(\frac{|x_n| + x_n}{2} \right) \right\} \right), \quad n = 0, 1, 2, \dots, \end{aligned} \quad (3.3)$$

where $\rho > 0$ and $\lambda > 0$ is a relaxation parameter.

Special Cases

If the nonlinear transformation A is zero, that is $A(u) = 0$, then Algorithms 3.1 and 3.3 reduce to the following algorithms for solving the complementarity problems (2.4) and (2.5), which are mainly due to Noor and Zarae [16].

ALGORITHM 3.4. Given $x_0 \in \mathbb{R}^n$, compute

$$\begin{aligned} x_{n+1} = & (1 - \lambda) \frac{|x_n| + x_n}{2} + \frac{\lambda}{2} \left(|x_n| + x_n - \rho E_n \left\{ M \left(\frac{|x_n| + x_n}{2} \right) + q \right. \right. \\ & \left. \left. + L_n(x_{n+1} - x_n) \right\} \right), \quad n = 0, 1, 2, \dots \end{aligned} \quad (3.4)$$

ALGORITHM 3.5. For given $x_0 \in \mathbb{R}^n$, compute

$$x_{n+1} = (1 - \lambda) \frac{|x_n| + x_n}{2} + \frac{\lambda}{2} \left(|x_n| + x_n - \rho \left\{ T \left(\frac{|x_n| + x_n}{2} \right) \right\} \right), \quad n = 0, 1, 2, \dots \quad (3.5)$$

Concerning the convergence of the Algorithms 3.4 and 3.5, Noor and Zarae [16] established a general convergence result for both the symmetric and nonsymmetric matrix M . We now study those conditions under which the approximate solution $\{x_n\}$ generated by the Algorithms 3.4 and 3.5 converges to the exact solution.

In brief, Algorithms 3.1 and 3.3 proposed in this paper are more general and include the previous ones as special case.

4. CONVERGENCE ANALYSIS

Here we study the convergence of the iterative $\{x_n\}$ generated by Algorithms 3.2 and 3.3. As special cases, we get the previously known results for the linear and nonlinear complementarity problems. For this purpose, we need the following concepts.

DEFINITION 4.1. A real matrix $M \in \mathbb{R}^{n \times n}$ is said to be Z -matrix (P -Matrix), if it has non-positive off-diagonal entries (positive principal minors). A square matrix with non-positive off-diagonal entries and with a non-negative inverse is called an M -matrix. It can be shown that a matrix which is both a Z -matrix and P -matrix is an M -matrix, see [17] for full details.

DEFINITION 4.2. A mapping $T: \mathbb{R}^n \rightarrow \mathbb{R}^n$ is said to be:

- (i) *Strongly monotone*, if there exists a constant $\alpha > 0$ such that

$$(Tu - Tv, u - v) \geq \alpha \|u - v\|^2, \quad \text{for all } u, v \in \mathbb{R}^n.$$

- (ii) *Lipschitz continuous*, if there exists a constant $\beta > 0$ such that

$$\|Tu - Tv\| \leq \beta \|u - v\|^2, \quad \text{for all } u, v \in \mathbb{R}^n.$$

In particular, it follows that $\alpha \leq \beta$.

We now state and prove the main results of this section.

THEOREM 4.1. *Suppose that there exists a non-negative matrix $N \in \mathbb{R}^{n \times n}$ such that*

$$|A(u) - A(v)| \leq N |u - v|, \quad \text{for all } u, v. \quad (4.1)$$

If $\{x_{n+1}\}$ and $\{x_n\}$ are the sequence generated by Algorithm 3.2, then

$$|x_{n+1} - x_n| \leq (2I - \lambda \rho E |L|)^{-1} [|2I - \lambda \rho E(M - L)| + \lambda \rho EN] |x_n - x_{n-1}|, \quad (4.2)$$

and

$$|x_{n+1} - x| \leq (2I - \lambda \rho E |L|)^{-1} [\lambda \rho EN + |2I - \lambda \rho E(M - L)|] |x_n - x_{n-1}|, \quad (4.3)$$

for each n , $x = u - \frac{1}{2}\rho E v$ and (u, v) is the solution of the mildly nonlinear complementarity problem (2.6). Here L is either a strictly lower or upper triangular matrix.

Proof. From the iterative scheme (3.2), we have

$$\begin{aligned} x_{n+1} - x_n &= (1 - \lambda) \frac{|x_n| - |x_{n-1}| + x_n - x_{n-1}}{2} \\ &\quad + \frac{\lambda}{2} \left((|x_n| - |x_{n-1}| + x_n - x_{n-1}) \right. \\ &\quad \left. - \rho E \left\{ M \left(\frac{|x_n| - |x_{n-1}| + x_n - x_{n-1}}{2} \right) \right. \right. \\ &\quad \left. \left. + L(x_{n+1} - x_n) - L(x_n - x_{n-1}) \right. \right. \\ &\quad \left. \left. + A \left(\frac{|x_n| + x_n}{2} \right) - A \left(\frac{|x_{n-1}| + x_{n-1}}{2} \right) \right\} \right). \end{aligned}$$

Taking absolute values of both sides, using (4.1), and the inequalities for $a, b \in \mathbb{R}^n$,

$$|a - b| \leq |a| + |b| \quad \text{and} \quad ||a| - |b|| \leq |a - b|,$$

we obtain

$$\begin{aligned} |x_{n+1} - x_n| &\leq (1 - \lambda) |x_n - x_{n-1}| + \lambda \left| I - \frac{\lambda \rho E}{2} (M - L) \right| |x_n - x_{n-1}| \\ &\quad + \frac{\lambda \rho E}{2} |L| |x_{n+1} - x_n| + \frac{\lambda \rho E}{2} N |x_n - x_{n-1}| \end{aligned}$$

which can be written as

$$(2I - \lambda\rho E |L|) |x_{n+1} - x_n| \leq [|2I - \lambda\rho E(M - L)| + \lambda\rho EN] |x_n - x_{n-1}|.$$

Since L is strictly upper or lowertriangular matrix, then $(2I - \lambda\rho E |L|)$ is an M -matrix. Thus $(2I - \lambda\rho E |L|)^{-1}$ exists and is non-negative. Hence we obtain

$$|x_{n+1} - x_n| \leq (2I - \lambda\rho E |L|)^{-1} [\lambda\rho EN + |2I - \lambda\rho E(M - L)|] |x_n - x_{n-1}|,$$

the required (4.2). Similarly, we can prove (4.3).

From Theorem 4.1, we can establish a condition for the convergence of the sequence $\{x_n\}$ of Algorithm 3.2 to be bounded and hence have an accumulation point, which is the solution of (2.6) and this is the main motivation of our next result.

THEOREM 4.2. *Assume that*

$$\sigma(G) < 1,$$

where

$$G = (2I - \lambda\rho E |L|)^{-1} [|2I - \lambda\rho E(M - L)| + \lambda\rho EN], \quad (4.4)$$

with σ denoting the spectral radius and $\lambda > 0$ is the relaxation parameter. Then for any initial vector x , the sequence $\{x_n\}$ generated by the iterative scheme (3.2) converges to a solution of (2.6).

Proof. The method of proof is similar to that used by Ahn [8], Pang [10], and Noor and Zarae [18]. First of all, we note that the matrix G defined by (4.4), is non-negative. From the relation, (4.1) and (4.2), we have

$$|x_{n+1} - x_n| \leq G |x_n - x_{n-1}|.$$

Since $\sigma(G) < 1$, it follows that, see Ortega and Rheinbold [19]

$$\lim_{n \rightarrow \infty} |x_{n+1} - x_n| = 0. \quad (4.5)$$

Next, by the inductive argument, we deduce that

$$\begin{aligned} |x_{n+1} - x_0| &\leq \sum_{i=0}^n G^i |x_1 - x_0| \\ &\leq (I - G)^{-1} |x_1 - x_0|, \end{aligned}$$

where the last inequality follows from the fact that the matrix G is non-negative and $\sigma(G) < 1$. This implies that the sequence $\{x_{n+1}\}$ is bounded and thus has an accumulation point x^* . Let $\{x_{n_i+1}\}$ be a sequence converging to x^* . Then from (4.5), we see that $\{x_{n_i+1}\}$ converges to x^* as well. Since the mappings T and A are continuous, so by passing to the limit $n_{i+1} \rightarrow \infty$, in the conditions

$$u_{n_i+1} \geq 0, \quad (Tu_{n_i+1} + A(u_{n_i+1})) \geq 0 \quad (u_{n_i+1}, Tu_{n_i+1} + A(u_{n_i+1})) = 0,$$

we deduce that x^* is a solution to the problem (2.9). Using the relations (2.6), and (2.8). By applying Theorem 4(ii) again, we obtain

$$|x_{n+1} - x^*| \leq G |x_n - x^*|,$$

where G is defined by (4.4). Since $\sigma(G) < 1$, it follows that the entire sequence $\{x_{n+1}\}$ converges to x^* , which is the required result.

From the proof of Theorem 4.2, it is clear that the condition $\sigma(G) < 1$ also provides existence and a uniqueness result for the mildly nonlinear complementarity problem (2.9) or, equivalently, to the problem (2.4). Theorem 4.2 is the main result of this subsection. Note also that our results hold for both the symmetric and non-symmetric matrix M . It is also worth mentioning that the convergence of the approximate solution $\{x_n\}$ obtained from (3.2) to the exact solution depends on the relaxation parameter $\lambda > 0$. For the strongly nonlinear complementarity problem, we prove the following result.

It can be shown, see Pang [10], that the condition (4.1) is equivalent to the fact that the nonlinear transformation A is Lipschitz continuous, i.e., there exists a constant $\gamma > 0$ such that

$$\|A(u) - A(v)\| \leq \gamma \|u - v\|, \quad \text{for all } u, v \in \mathbb{R}^n. \quad (4.6)$$

THEOREM 4.3. *Let the mapping T from \mathbb{R}^n into itself be strongly monotone and Lipschitz continuous. If the nonlinear transformation A is also Lipschitz continuous with Lipschitz constant γ such that $\gamma < \alpha$, where α is the strongly monotonicity constant of T , x and x_{n+1} are the solutions of (2.10) and (3.3), then*

$$x_{n+1} \rightarrow x \quad \text{in } \mathbb{R}^n,$$

for $0 < \rho < 4(\alpha - \gamma)/(\beta^2 - \gamma^2)$, $\rho\gamma < 1$, and $\lambda < 2/(2 - \rho\gamma - \sqrt{4 - 4\rho\gamma - \beta^2\rho^2})$, where β is defined in Definition 4.2.

Proof. From (2.10) and (3.3), we obtain

$$\begin{aligned}
 x_{n+1} - x &= (1 - \lambda) \frac{|x_n| + x_n}{2} \\
 &\quad + \lambda \left(\frac{|x_n| + x_n}{2} - \frac{\rho}{2} \left\{ T \left(\frac{|x_n| + x_n}{2} \right) + A \left(\frac{|x_n| + x_n}{2} \right) \right\} \right) \\
 &\quad - (1 - \lambda) \frac{|x| + x}{2} \\
 &\quad - \left(\frac{|x| + x}{2} - \frac{\rho}{2} \left\{ T \left(\frac{|x| + x}{2} \right) + A \left(\frac{|x| + x}{2} \right) \right\} \right) \\
 \|x_{n+1} - x\| &\leq (1 - \lambda) \|x_n - x\| + \lambda \left\| \frac{|x_n| + x_n - |x| - x}{2} \right. \\
 &\quad \left. - \frac{\rho}{2} \left\{ T \left(\frac{|x_n| + x_n}{2} \right) \right. \right. \\
 &\quad \left. \left. - T \left(\frac{|x| + x}{2} \right) + A \left(\frac{|x_n| + x_n}{2} \right) - A \left(\frac{|x| + x}{2} \right) \right\} \right\| \\
 &\leq (1 - \lambda) \|x_n - x\| + \frac{\lambda \rho}{2} \left\| A \left(\frac{|x_n| + x_n}{2} \right) - A \left(\frac{|x| + x}{2} \right) \right\| \\
 &\quad + \lambda \left\| \frac{|x_n| + x_n - |x| - x}{2} \right. \\
 &\quad \left. - \frac{\rho}{2} \left\{ T \left(\frac{|x_n| + x_n}{2} \right) - T \left(\frac{|x| + x}{2} \right) \right\} \right\|. \tag{4.7}
 \end{aligned}$$

Now by the strongly monotonicity and Lipschitz continuity of T , we have

$$\begin{aligned}
 &\left\| \frac{|x_n| + x_n - |x| - x}{2} - \frac{\rho}{2} \left\{ T \left(\frac{|x_n| + x_n}{2} \right) - T \left(\frac{|x| + x}{2} \right) \right\} \right\|^2 \\
 &\leq \left(1 - \alpha \rho + \frac{\rho^2 \beta^2}{4} \right) \|x_n - x\|^2. \tag{4.8}
 \end{aligned}$$

From (4.7), and (4.8), we obtain

$$\begin{aligned}
 \|x_{n+1} - x\| &\leq \left(1 - \lambda + \frac{\lambda \rho \gamma}{2} + \gamma \sqrt{1 - \alpha \rho + \frac{\rho^2 \beta^2}{4}} \right) \|x_n - x\| \\
 &= \theta \|x_n - x\|,
 \end{aligned}$$

where $\theta = (1 - \lambda + \lambda\rho\gamma/2 + \lambda\sqrt{1 - \alpha\rho + \rho^2\beta^2/4}) < 1$ for $0 < \rho < 4(\alpha - \gamma)/(\beta^2 - \gamma^2)$, $\rho\gamma < 1$, and $\lambda \leq 2/(2 - \rho\gamma - (4 - 4\alpha\rho + \beta^2\beta^2)^{1/2})$.

Since $\theta < 1$, then the fixed-point problem (2.10) has a unique solution and consequently the iterates x_{n+1} obtained from (3.3) converge to the solution x of (2.10), which is the required result.

From the proof of Theorem 4.3, it is clear that the convergence criteria depends on the relaxation parameter $\lambda > 0$ for strongly nonlinear complementarity problems. We also note that if the non-linear transformation A is independent of the solution u or identically equal to zero, then Theorem 4.3 is exactly the same as proved for nonlinear complementarity problem by Noor and Zaraf [16].

5. CONCLUSION

In this paper, using the technique of change of variables, we have characterized the complementarity problems in terms of the fixed-point problems. We have only shown the possibility that the fixed-point formulation enables us to suggest and analyze new, more general and unified algorithms for the complementarity problems in a natural and elegant way. We have also studied those conditions under which the approximate solutions obtained from these iterative algorithms converge to the exact solution. It is shown that the convergence criteria is compatible with the previous known algorithms of Mangasarian, see Ahn [8, 9]. Development of implementable algorithms needs further research efforts.

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